

Ring Theoretic Properties of Partial Crossed products and related themes ¹

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Abstract

In this paper we work with unital twisted partial actions. We investigate ring theoretic properties of partial crossed products as artinianity, noetherianity, perfect property, semilocalproperty, semiprimary property and we also study the Krull dimension. Moreover, we consider triangular matrix representation of partial skew group rings, weak and global dimensions of partial crossed products Also we study when the partial crossed products are Frobenius and symmetric algebras.

Introduction

Partial actions of groups have been introduced in the theory of operator algebras as a general approach to study C^* -algebras by partial isometries (see, in particular, [15] and [16]), and crossed products classically, as well-pointed out in [12], are the center of the rich interplay between dynamical systems

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and operator algebras (see, for instance, [22] and [27]). The general notion of (continuous) twisted partial action of a locally compact group on a C^* -algebra and the corresponding crossed product were introduced in [15]. Algebraic counterparts for some notions mentioned above were introduced and studied in [10], stimulating further investigations, see for instance, [2], [4], [18] and references therein. In particular, twisted partial actions of groups on abstract rings and corresponding crossed products were recently introduced in [12].

In [3] and [7] the authors investigated ring theoretic properties of partial crossed products. In this article, we continue the investigation of ring theoretic properties of partial crossed products as artinianity, noetherianity, semilocal property, perfect property and semiprimary property. In these cases, we study necessary and sufficient conditions for the partial crossed products satisfy such conditions. We study the Krull dimension of partial crossed products and we compare it with the Krull dimension of the base ring.

We study triangular matrix representation of partial skew group rings and we study global dimension, weak global dimension of partial crossed products. Moreover, we investigate when the partial crossed products are Frobenius and symmetric algebras.

This article is organized as follows: In the Section 1, we give some preliminaries and results that will be used during this paper.

In the Section 2, we study necessary and sufficient conditions for the partial crossed products to be artinian, noetherian, semilocal, semiprimary and perfect, where we generalize the results presented in [25] and [26].

In the Section 3, we only consider partial actions of groups and we study the Krull dimension of partial skew group rings and we compare it with the Krull dimension of the base ring. In these cases we extend the results presented in [26].

In the Section 4, we define relative partial actions of groups on bimodules and triangular matrix algebras, and we give some conditions for the existence of enveloping actions. Moreover, we study triangular matrix representation of partial skew group rings.

In the Section 5, we study global dimension, weak global dimension of partial crossed products. We finish the article by presenting conditions for

partial crossed products to be Frobenius and symmetric algebras.

1 Preliminaries

In this section, we recall some notions about twisted partial actions on rings. More details can be found in [12], [13] and [10].

We begin with the following definition that is a particular case of ([13], Definition 2.1).

Definition 1. *An unital twisted partial action of a group G on a ring R is a triple*

$$\alpha = (\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G}, \{w_{g,h}\}_{(g,h) \in G \times G}),$$

where for each $g \in G$, D_g is a two-sided ideal in R generated by a central idempotent 1_g , $\alpha_g : D_{g^{-1}} \rightarrow D_g$ is an isomorphism of rings and for each $(g, h) \in G \times G$, $w_{g,h}$ is an invertible element of $D_g D_{gh}$, satisfying the following postulates, for all $g, h, t \in G$:

- (i) $D_e = R$ and α_e is the identity map of R ;
- (ii) $\alpha_g(D_{g^{-1}} D_h) = D_g D_{gh}$;
- (iii) $\alpha_g \circ \alpha_h(a) = w_{g,h} \alpha_{gh}(a) w_{g,h}^{-1}$, for all $a \in D_{h^{-1}} D_{h^{-1}g^{-1}}$;
- (iv) $w_{g,e} = w_{e,g} = 1$;
- (v) $\alpha_g(aw_{h,t})w_{g,ht} = \alpha_g(a)w_{g,h}w_{gh,t}$, for all $a \in D_{g^{-1}} D_h D_{ht}$.

Remark 2. *If $w_{g,h} = 1_g 1_{gh}$, for all $g, h \in G$, then we have a partial action as defined in ([10], Definition 1.1) and when $D_g = R$, for all $g \in G$, we have that α is a twisted global action.*

Let $\beta = (T, \{\beta_g\}_{g \in G}, \{u_{g,h}\}_{(g,h) \in G \times G})$ be a twisted global action of a group G on a (non-necessarily unital) ring T and R an ideal of T generated by a central idempotent 1_R . We can restrict β to R as follows. Putting $D_g = R \cap \beta_g(R) = R \cdot \beta_g(R)$, $g \in G$, each D_g has an identity element $1_R \beta_g(1_R)$. Then defining $\alpha_g = \beta_g|_{D_{g^{-1}}}$, for all $g \in G$, the items (i), (ii) and (iii) of Definition 1 are satisfied. Furthermore, defining $w_{g,h} = u_{g,h} 1_R \beta_g(1_R) \beta_{gh}(1_R)$,

$g, h \in G$, the items (iv), (v) e (vi) of Definition 1 are also satisfied. So, we obtain a twisted partial action of G on R .

The following definition appears in ([13], Definition 2.2).

Definition 3. *A twisted global action $(T, \{\beta_g\}_{g \in G}, \{u_{g,h}\}_{(g,h) \in G \times G})$ of a group G on an associative (non-necessarily unital) ring T is said to be an enveloping action (or a globalization) for a twisted partial action α of G on a ring R if there exists a monomorphism $\varphi : R \rightarrow T$ such that, for all g and h in G :*

- (i) $\varphi(R)$ is an ideal of T ;
- (ii) $T = \sum_{g \in G} \beta_g(\varphi(R))$;
- (iii) $\varphi(D_g) = \varphi(R) \cap \beta_g(\varphi(R))$;
- (iv) $\varphi \circ \alpha_g(a) = \beta_g \circ \varphi(a)$, for all $a \in D_{g^{-1}}$;
- (v) $\varphi(aw_{g,h}) = \varphi(a)u_{g,h}$ and $\varphi(w_{g,h}a) = u_{g,h}\varphi(a)$, for all $a \in D_g D_{gh}$.

In ([13], Theorem 4.1), the authors studied necessary and sufficient conditions for a twisted partial action α of a group G on a ring R has an enveloping action. Moreover, they studied which rings satisfy such conditions.

Suppose that (R, α, w) has an enveloping action (T, β, u) . In this case, we may assume that R is an ideal of T and we can rewrite the conditions of the Definition 3 as follows:

- (i') R is an ideal of T ;
- (ii') $T = \sum_{g \in G} \beta_g(R)$;
- (iii') $D_g = R \cap \beta_g(R)$, for all $g \in G$;
- (iv') $\alpha_g(a) = \beta_g(a)$, for all $x \in D_{g^{-1}}$ and $g \in G$;
- (v') $aw_{g,h} = au_{g,h}$ and $w_{g,h}a = u_{g,h}a$, for all $a \in D_g D_{gh}$ and $g, h \in G$.

Given a twisted partial action α of a group G on a ring R , we recall from ([12], Definition 2.2) that the *partial crossed product* $R *_{\alpha, w} G$ is the direct sum

$$\bigoplus_{g \in G} D_g \delta_g,$$

where δ_g 's are symbols, with the usual addition and multiplication defined by

$$(a_g \delta_g)(b_h \delta_h) = \alpha_g(\alpha_g^{-1}(a_g)b_h)w_{g,h}\delta_{gh}.$$

By ([12], Theorem 2.4), $R *_{\alpha,w} G$ is an associative ring whose identity is $1_R \delta_1$. Note that we have the injective morphism $\phi : R \rightarrow R *_{\alpha,w} G$, defined by $r \mapsto r \delta_1$ and we can consider $R *_{\alpha,w} G$ as an extension of R .

The following definition appears in ([3], Definition 4.13).

Definition 4. *Let α be an unital twisted partial action. We say that α is of finite type if there exists a finite subset $\{g_1, g_2, \dots, g_n\}$ of G such that*

$$\sum_{i=1}^n D_{gg_i} = R,$$

for all $g \in G$.

It is convenient to point out that in the same way as in ([18], Proposition 1.2), we can prove that an unital twisted partial action α of a group G on an unital ring R with an enveloping action (T, β, u) is of finite type if and only if there exists $g_1, \dots, g_n \in G$ such that $T = \sum_{i=1}^n \beta_{g_i}(R)$ if and only if T has an identity element.

It is convenient to remind that two rings A, B are Morita equivalent rings if their module categories are equivalent, see ([1], pg 251), for more details. Now, we have the following result which is a particular case of ([13], Theorem 3.1).

Proposition 5. *Let α be an unital twisted partial action of a group G on a ring R with enveloping action (T, β, u) . If α is of finite type, then the rings $R *_{\alpha,w} G$ and $T *_{\beta,u} G$ are Morita equivalent rings.*

Let α be a twisted partial action of a group G on a ring R . A subring (resp. ideal) S of R is said to be α -invariant if $\alpha_g(S \cap D_{g^{-1}}) \subseteq S \cap D_g$, for all $g \in G$. This is equivalent to say that $\alpha_g(S \cap D_{g^{-1}}) = S \cap D_g$, for all $g \in G$. In this case the set

$$S *_{\alpha,w} G = \left\{ \sum_{g \in G} a_g \delta_g \mid a_g \in S \cap D_g \right\}$$

is a subring (resp. ideal) of $R *_{\alpha,w} G$.

If I is an α -invariant ideal of R , then the twisted partial action α can be extended to an twisted partial action $\bar{\alpha}$ of G on R/I as follows: for each $g \in G$, we define $\bar{\alpha}_g : D_{g^{-1}} + I \longrightarrow D_g + I$, putting $\bar{\alpha}_g(a + I) = \alpha_g(a) + I$, for all $a \in D_{g^{-1}}$, and for each $(g, h) \in G \times G$, we extend each $w_{g,h}$ to R/I by

$\overline{w}_{g,h} = w_{g,h} + I$. In this case, $I *_{\alpha,w} G$ is an ideal of $R *_{\alpha,w} G$ and there exists a natural isomorphism between $(R *_{\alpha,w} G)/(I *_{\alpha,w} G)$ and $(R/I) *_{\overline{\alpha},\overline{w}} G$. Moreover, when (R, α, w) has enveloping action (T, β, u) , then by similar methods presented in Section 2 of [18], we have that $(T/I^e, \overline{\beta}, \overline{u})$ is the enveloping action of $(R/I, \overline{\alpha}, \overline{w})$, where I^e is the β -invariant ideal such that $I^e \cap R = I$. Following ([18], Remark 2.3]) we have that the Jacobson radical $J(R)$ of R is α -invariant and so, we can extend α to $R/J(R)$.

For convenience, we recall some concepts of ring theory that we will freely use in this paper (see [20] for more details).

Definition 6. *Let S be a ring. Then we have the following definitions.*

- (i) *S is semiprimitive if the Jacobson radical $J(S)$ is zero.*
- (ii) *S is semilocal if $S/J(S)$ is semisimple.*
- (iii) *S is right perfect if S is semilocal and $J(S)$ is T -nilpotent, or equivalently, S is right perfect if S satisfies DCC on principal left ideals (see ([20], Theorem 23.20)).*
- (iv) *S is semiprimary if S is semilocal and $J(S)$ is nilpotent.*

We finish this section with the following three results, where the first and second result appear in ([24], Lemma 3) and ([25], Proposition 2.1), respectively.

Proposition 7. *Let S be a ring with identity and B a subring of S with the same identity such that B is a left B -direct summand of S . If S is semilocal then so is B .*

Proposition 8. *Let S be a ring with identity and B a subring of S with the same identity. If B is a left (resp. right) B -direct summand of S , then for any right (resp. left) ideal I of B , $IS \cap B = I$ (resp. $SI \cap B = I$).*

As a particular case of Proposition 8 we have the following lemma.

Lemma 9. *Let α be an unital twisted partial action of a group G on R and B an α -invariant subring of R such that ${}_B R = {}_B B \oplus {}_B C$. Then for any right ideal I of $B *_{\alpha,w} G$ we have that $I(R *_{\alpha,w} G) \cap (B *_{\alpha,w} G) = I$.*

2 Ring Theoretic Properties of Partial Crossed Products

Throughout this section R is an associative ring with an identity element 1_R , G is a group and $\alpha = (\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G}, \{w_{g,h}\}_{(g,h) \in G \times G})$ is an unital twisted partial action of G on R such that α does not necessarily admit enveloping action, unless otherwise stated.

Let $x = \sum a_g u_g \in R *_{\alpha,w} G$. Then the *support* of x is the set

$$\text{supp}(x) = \{g \in G \mid a_g \neq 0\}.$$

For each subgroup H of G , we can restrict α to H , i.e., if we define (R, α_H, w_H) as $\alpha_H = \{\alpha_h : D_{h^{-1}} \rightarrow D_h \mid h \in H\}$ and $w_H = \{w_{l,m}\}_{l,m \in H}$ then we have a twisted partial of H on R . Moreover, we have the partial crossed product

$$R *_{\alpha_H, w_H} H = \{\sum_{h \in H} a_h \delta_h \mid a_h \in D_h\}$$

with usual sum and multiplication rule $(a_h \delta_h)(b_l \delta_l) = \alpha_h(\alpha_{h^{-1}}(a_h) b_l) w_{h,l} \delta_{hl}$ and it is a subring of $R *_{\alpha,w} G$ with the same identity 1_R of $R *_{\alpha,w} G$.

Let A be the set

$$A = \{x \in R *_{\alpha,w} G \mid \text{supp}(x) \subseteq G - H\}.$$

Since $\text{supp}(0) = \emptyset \subseteq G - H$, we have that $0 \in A$ and obviously

$$A \cap (R *_{\alpha_H, w_H} H) = 0.$$

The following result will be useful in this section.

Lemma 10. *Let α be an unital twisted partial action of a group G on R and H a subgroup of G . Then $A = \{x \in R *_{\alpha,w} G \mid \text{supp}(x) \subseteq G - H\}$ is a left and right $(R *_{\alpha_H, w_H} H)$ -submodule of $R *_{\alpha,w} G$, where the sum and the scalar multiplication are those inherited from $R *_{\alpha,w} G$. Moreover, in this case $(R *_{\alpha_H, w_H} H)$ is a left and right $(R *_{\alpha_H, w_H} H)$ -direct summand of $R *_{\alpha,w} G$ and A is the complement.*

Proof. It is not difficult to show that A is an $(R *_{\alpha_H, w_H} H, R *_{\alpha_H, w_H} H)$ -bimodule. Finally, it is easy to see that $R *_{\alpha, w} G = (R *_{\alpha_H, w_H} H) \oplus A$ as a right $(R *_{\alpha_H, w_H} H)$ -module. The left side is completely similar. \square

Next, using Lemma 10 we easily get the following result and we give a short proof for reader's convenience.

Proposition 11. *Let α be an unital twisted partial action of a group G on a ring R and H a subgroup of G . If $R *_{\alpha, w} G$ is right artinian (resp. right noetherian, right perfect, semilocal), then $R *_{\alpha_H, w_H} H$ is right artinian (right noetherian, right perfect, semilocal).*

Proof. By Lemma 10, we have the canonical projection

$$\pi : R *_{\alpha, w} G \rightarrow R *_{\alpha_H, w_H} H.$$

So, we easily obtain that $R *_{\alpha_H, w_H} H$ is right artinian (right noetherian, right perfect, semilocal). \square

Remark 12. *If H is the trivial subgroup $\{1_G\}$ of G , then $R *_{\alpha_H, w_H} H = R$. So, by Lemma 10, R is a right (resp. left) R -direct summand of $R *_{\alpha, w} G$ and the complement A is given by*

$$A = \{\sum a_g u_g \in R *_{\alpha} G \mid g \neq 1_G\}.$$

As an immediate consequence of Proposition 11 we have the following result.

Corollary 13. *Let α be an unital twisted partial action of a group G on a ring R . If $R *_{\alpha, w} G$ is right artinian (right noetherian, right perfect, semilocal), then R is right artinian (right noetherian, right perfect, semilocal).*

To study necessary and sufficient conditions for the partial crossed product to be right artinian, we need the following three results, where we work with twisted global actions where we study artinianity in global crossed products.

It is convenient to remind that the *partial fixed ring* is

$$R^\alpha = \{x \in R \mid \alpha_g(x 1_{g^{-1}}) = x 1_g, \forall g \in G\}.$$

See [9] and [18] for more details.

Lemma 14. *Let β be a twisted global action of a group G on a primitive ring T . If $T *_{\beta,u} G$ is right artinian, then G is finite.*

Proof. By Corollary 13 we have that T is right artinian. Thus, T is a simple artinian ring and we have that $Z(T) = F$ is a field. Let F^β be the fixed field by β . Then, $F^\beta *_{\beta,u} G$ is right artinian since $F^\beta *_{\beta,u} G$ is a direct summand of $T *_{\beta,u} G$. Note that $F^\beta \subseteq K$, where K is an algebraically closed field and it is not difficult to show that we can extend β to $F^\beta \otimes K$ and consequently we have the isomorphisms $F^\beta *_{\beta,u} G \otimes K \simeq F^\beta \otimes K[G] \simeq K *_{\beta,u} G$ that is right artinian, where $K *_{\beta,u} G$ is a twisted group ring. Hence, the twisted group ring $K *_{\beta,u} G$ is right artinian and it has finite dimension as K -vector space. So, G is finite. \square

Lemma 15. *Let β be a twisted global action of a group G on a semiprimitive ring T . If $T *_{\beta,u} G$ is right artinian, then T is right artinian and G is finite.*

Proof. Suppose that $T *_{\beta,u} G$ is artinian. Then by Corollary 13 we have that T is right artinian and it follows that T is semiprimitive Artinian. Now, let $T = T_1 \oplus \cdots \oplus T_n$ be the decomposition in simple components and consider the subgroup $H = \{g \in G : \beta_g = id_T\}$ of G . Note that the twisted group ring $T_i *_{\beta,u} H$ is right Artinian, for all $i \in \{1, \dots, n\}$. Now, as in the Lemma 14 we have that H is finite. Hence, using the similar methods of the second paragraph in the proof of ([25], Lemma 3.2) we get that G is finite.

The converse is trivial. \square

Now, with the last two lemmas in mind, the proof of the next proposition follows with similar methods of ([25], Theorem 3.3).

Proposition 16. *Let β be a twisted global action on a ring T . Then $T *_{\beta,u} G$ is right artinian if and only if T is right artinian and G is finite.*

Now, we are in conditions to prove the first main result of this section that generalizes ([25], Theorem 3.3).

Theorem 17. *Let α be an unital twisted partial action of a group G on a ring R such that α is of finite type. Then $R *_{\alpha,w} G$ is right artinian if and only if R is right artinian and G is a finite group.*

Proof. By Corollary 13 we have that R is right artinian. Thus, $R/J(R)$ is semiprimitive artinian and it is semisimple by ([20], Proposition 11.7). Since $R/J(R) *_{\bar{\alpha}, \bar{w}} G$ is right artinian, then we may assume that R is a semisimple. In this case, α admits a globalization (T, β, u) and following the same steps of ([18], Corollary 1.3 and 1.8), T is a semiprimitive right artinian ring with identity. By Proposition 5 we have that $R *_{\alpha, w} G$ and $T *_{\beta, u} G$ are Morita equivalent and it follows from ([23], Lemma 3.5.8) that $T *_{\beta, u} G$ is right artinian. So, by Proposition 16 we have that G is finite

The converse is trivial. \square

Now, we present an example where the assumption “ α is of finite type” on Theorem 17 is not superfluous.

Example 18. Let $R = Ke_1 \oplus Ke_2$, where $\{e_1, e_2\}$ is a set of central orthogonal idempotents and K is a field. We define the action of $G = \mathbb{Z}$ as follows: $D_0 = R$, $D_{-1} = Ke_1$, $D_1 = Ke_2$, $D_i = 0$, for $i \neq 0, 1, -1$, $\alpha_0 = id_R$, $\alpha_1(e_1) = e_2$, $\alpha_{-1}(e_2) = e_1$ and $\alpha_i = 0$ for $i \neq 0, 1, -1$. We clearly have that $R *_{\alpha} G$ is a finite dimensional vector space and we obtain that $R *_{\alpha} G$ is right artinian with G an infinite group and α is not of finite type.

One may ask if we input the assumption $D_g \neq 0$, $g \in G$ and α is any twisted partial action would be enough to get the result of Theorem 15. The next example shows that this is not the case.

Example 19. Let $R = Ke_1 \oplus Ke_2$ and a partial action of \mathbb{Z} of R is defined as follows: $D_e = R$, $\alpha_e = id_R$, $D_i = Ke_1$ and $\alpha_i = id_{Ke_1}$ for all $i \neq 0$. Then $R *_{\alpha} G = R \oplus (\oplus_{i \neq 0} Ke_1 \delta_i)$. Note that in this case $R *_{\alpha} G$ is commutative and we only have the ideals $I_1 = Ke_1 \oplus (\oplus_{i \neq 0} Ke_1 \delta_i)$ and $I_2 = Ke_2 \oplus (\oplus_{i \neq 0} Ke_1 \delta_i)$. Thus, $R *_{\alpha} G$ is artinian and G is infinite.

In [4], the authors studied the noetherianity of partial skew groups rings with the assumption that the group is polycyclic-by-finite. In the next result we study the noetherianity of partial crossed products where the group is not necessarily polycyclic-by-finite.

Theorem 20. Let α be an unital twisted partial action of a group G on R such that only a finite number of ideals D_g are not the zero ideal. Then R is right noetherian if and only if $R *_{\alpha, w} G$ is right noetherian.

Proof. Suppose that R is right noetherian. Then each ideal D_g , $g \in G$, is finite generated (as a right R -module) and since D_g and $\delta_g D_g$ are R -isomorphic, we have that each $u_g D_g$ is finitely generated as a right R -module. Now, using the assumption we have that $R *_{\alpha,w} G = \bigoplus_{g \in G} \delta_g D_g$ is finitely generated as a right R -module. So, by ([23], Lemma 1.1.3), $R *_{\alpha,w} G$ is right noetherian. The converse follows from Corollary 13. \square

As a particular case of Theorem 20 we have the following result.

Corollary 21. *Let α be an unital twisted partial action of a finite group G on R . Then R is right noetherian if and only if $R *_{\alpha,w} G$ is right noetherian.*

According to ([28], pg. 1) the class of groups \mathcal{H} is defined by the groups that are either finite or contains an infinite abelian subgroup. Now, we are in conditions to prove the following result that generalizes ([25], Theorem 3.7)

Theorem 22. *Let α be an unital twisted partial action of a group $G \in \mathcal{H}$ on R such that α is of finite type. Then $R *_{\alpha,w} G$ is right perfect if and only if R is right perfect and G is finite.*

Proof. Suppose that $R *_{\alpha,w} G$ is right perfect. By the fact that α is of finite type, we have by Proposition 5 that $R *_{\alpha,w} G$ and $T *_{\beta,u} G$ are Morita equivalent rings then we get that $T *_{\beta,u} G$ is right perfect by ([1], Corollary 28.6). Note that, by Corollary 13 and ([20], Corollary 24.19), T and $(T/J(T)) *_{\bar{\beta},\bar{u}} G \cong (T *_{\beta,u} G)/(J(T) *_{\beta,u} G)$ are right perfect. Thus, we may assume that T is semisimple and let $H = \{h \in G : \beta_h(x) = x, \forall x \in T\}$. Then H is a subgroup of G and by Proposition 11 we have that $T *_{\beta,u} H$ is a perfect twisted group ring. Hence, by ([28], Theorem 2.6) we have that H is finite. Now, following the same arguments applied to the artinian case in ([25], Lemma 3.2) we get that G/H is finite. So, G is finite.

Conversely, assume that R is right perfect and G is finite. The proof follows the same arguments of ([25], Theorem 3.7). \square

If α is not of finite type, the result of the last theorem is not true as the next example shows.

Example 23. *Let $R = K$ and $G = \mathbb{Z}$. We define the following partial action: $D_0 = R$ and $\alpha_0 = id_K$, $D_s = 0$ and $\alpha_s = 0$ for all $s \neq 0$. Thus, $R *_{\alpha} G = R$ that is right perfect, with G an infinite group, and α is not of finite type.*

In the next result, we study the perfect, semiprimary and semilocal property between a ring R with a twisted partial action α and (T, β, u) its enveloping action when it exists.

Proposition 24. *Let α be an unital twisted partial action of a group G on a ring R such that α has an enveloping action (T, β, u) . The following statements hold.*

- (i) *If R is right perfect and α is of finite type, then T is right perfect.*
- (ii) *If T is right perfect, then R is right perfect.*
- (iii) *If R is semilocal (semiprimary) and α is of finite type, then T is semilocal (semiprimary).*
- (iv) *If T is semilocal (semiprimary) then R is semilocal (semiprimary).*

Proof. (i) Suppose that R is right perfect and consider $Tt_1 \supseteq Tt_2 \supseteq \cdots$ a descending chain of principal left ideals of T . Since α is of finite type then there are $g_1, \dots, g_n \in G$ such that $T = \beta_{g_1}(R) + \cdots + \beta_{g_n}(R)$. Thus, for each $i = 1, 2, \dots, n$, we have

$$\beta_{g_i}(1_R)Tt_1 \supseteq \beta_{g_i}(1_R)Tt_2 \supseteq \cdots$$

Now for each $i = 1, 2, \dots, n$ and $j \geq 1$,

$$\begin{aligned} x \in \beta_{g_i}(1_R)Tt_j &\Leftrightarrow x = \beta_{g_i}(1_R)tt_j, \quad t \in T \\ &\Leftrightarrow x = t\beta_{g_i}(1_R)t_j, \quad t \in T \\ &\Leftrightarrow x = t\beta_{g_i}(1_R)[\beta_{g_i}(1_R)t_j], \quad t \in T \\ &\Leftrightarrow x = y\beta_{g_i}(1_R)t_j, \quad y \in \beta_{g_i}(R) \\ &\Leftrightarrow x \in \beta_{g_i}(R)[\beta_{g_i}(1_R)t_j] \end{aligned}$$

Hence, for each $i = 1, 2, \dots, n$ and $j \geq 1$, $\beta_{g_i}(1_R)Tt_j$ is an principal left ideal of $\beta_{g_i}(R)$. Since each $\beta_{g_i}(R)$ is right perfect, $i = 1, 2, \dots, n$, there exist m_i such that

$$\beta_{g_i}(1_R)Tt_{m_i} = \beta_{g_i}(1_R)Tt_{m_i+1} = \cdots$$

Let $m = \max\{m_i \mid 1 \leq i \leq n\}$. Then we obtain that

$$\beta_{g_i}(1_R)Tt_m = \beta_{g_i}(1_R)Tt_{m+1} = \cdots,$$

for each $i = 1, 2, \dots, n$. Following ([18], Observation 1.13(iv)), we obtain $Tt_m = Tt_{m+1} = \cdots$. Finally, by the fact that α is of finite type we have that

the extension of α to $R/J(R)$ is of finite type. So, using the same methods of ([18], Corollary 1.3) we have that $T/J(T)$ is artinian. Therefore, T is perfect.

(ii) Suppose that T is right perfect and consider

$$Ra_1 \supseteq Ra_2 \supseteq \cdots$$

a chain of principal left ideals of R . Following ([18], Observation 1.13 (ii)), this is also a chain of principal left ideals of T . Since T is right perfect, there exists m such that $Ra_m = Ra_{m+1} = \cdots$. Using the same arguments presented in [18], we have that $(T/J(T), \overline{\beta}, \overline{u})$ is the enveloping action of $(R/J(R), \overline{\alpha}, \overline{w})$, where $\overline{\beta}$ and $\overline{\alpha}$ are the extensions of β and α to $T/J(T)$ and $R/J(R)$, respectively. Since $T/J(T)$ is artinian, then by the same methods of ([18], Proposition 1.2 and Corollary 1.3) we get that $R/J(R)$ is right artinian. So, R is right perfect.

(iii) We only prove for the semilocal case, because the semiprimary case is similar. By assumption, $R/J(R)$ is semisimple artinian, and using the methods presented in [18] we have that $(T/J(T), \overline{\beta}, \overline{u})$ is the enveloping action of $(R/J(R), \overline{\alpha}, \overline{w})$. Since $R/J(R)$ is artinian and the extension $\overline{\alpha}$ of α to $R/J(R)$ is of finite type, then by ([18], Corollary 1.3) we have that $T/J(T)$ is artinian. Hence, $T/J(T)$ is semisimple artinian. So, T is semilocal.

(iv) As in item (iii), we only prove for the semilocal case. As mentioned in item (iii) we have that $(T/J(T), \overline{\beta}, \overline{u})$ is the enveloping action of $(R/J(R), \overline{\alpha}, \overline{w})$. Since $R/J(R)$ is an ideal of $T/J(T)$ then $R/J(R)$ is artinian. By the fact that $R/J(R)$ is semiprimitive we have that $R/J(R)$ is semisimple. So, R is semilocal. \square

In general R is right perfect does not imply that T is right perfect as the next example shows which implies that the assumption of “ α is of finite type” in item (i) above can not be dropped.

Example 25. Let $T = \oplus_{i \in \mathbb{Z}} Ke_i$, where $\{e_i : i \in \mathbb{Z}\}$ is a set of central orthogonal idempotents, K is a field, and $R = Ke_0$. We define an action of an infinite cyclic group G generated by σ as follows: $\sigma(e_i) = e_{i+1}$, for all $i \in \mathbb{Z}$. Thus we easily have that an induced partial action of G on R . Note that R is right perfect, but T is not right perfect.

Theorem 26. *Let α be an unital twisted partial action of a group G on a ring R and (T, β, u) its enveloping action. Suppose that $G \in \mathcal{H}$ and α is of finite type. Then the following conditions are equivalent:*

- (i) *R is right perfect and G is finite.*
- (ii) *T is right perfect and G is finite.*
- (iii) *$T *_{\beta, u} G$ is right perfect.*
- (iv) *$R *_{\alpha, w} G$ is right perfect.*

Proof. By Proposition 24 we have that (i) and (ii) are equivalent. By the fact that α is of finite type we have by Proposition 5 that $T *_{\beta, u} G$ and $R *_{\alpha, w} G$ are Morita equivalent and using ([1], Corollary 28.6) we easily have that (iii) and (iv) are equivalent. Moreover, by Theorem 22, we have that (i) and (iv) are equivalent. \square

Remark 27. *It is convenient to point out that Theorem 26 holds without the assumption that $G \in \mathcal{H}$ if the twisted partial action is simply a partial action*

In the next two results we study necessary and sufficient conditions for the partial crossed product to be semiprimary and semilocal with the assumption that the group is in the class \mathcal{H} and these generalize ([25], Theorem 3.8), ([25], Proposition 4.2) and ([24], Corollary 2).

Theorem 28. *Let α be an unital twisted partial action of a group G on a ring R such that $G \in \mathcal{H}$. Then $R *_{\alpha, w} G$ is semiprimary if and only if R is semiprimary and G is a finite group.*

Proof. Suppose that $R *_{\alpha, w} G$ is semiprimary. Then $R *_{\alpha, w} G$ is right perfect and by Theorem 22, G is finite and R is right perfect. Hence, $R/J(R)$ is semisimple. Since $J(R *_{\alpha} G)$ is nilpotent (because $R *_{\alpha, w} G$ is semiprimary), then following ([18], Corollary 6.5(ii)), we have that $J(R)$ is nilpotent. So, R is semiprimary.

Conversely, suppose that R is semiprimary and G is finite. Then $\bar{R} = R/J(R)$ is semisimple and, consequently, it is right artinian. Hence following ([18], Corollary 6.5(ii)), $J(R *_{\alpha, w} G)$ is nilpotent. Note that by Theorem 22,

we have that $R *_{\alpha,w} G$ is right perfect. So, $R *_{\alpha,w} G$ is semilocal and we have that $R *_{\alpha,w} G$ is semiprimary. \square

Theorem 29. *Let α be an unital twisted partial action of a finite group G on a ring R . Then R is semilocal if and only if $R *_{\alpha,w} G$ is semilocal.*

Proof. If $R *_{\alpha,w} G$ is semilocal, then by Corollary 13, R is semilocal. Conversely, suppose that R is semilocal. Then $\bar{R} = R/J(R)$ is semisimple. Hence $R/J(R)$ is right artinian and using Theorem 17 and ([7], Lemma 3.8) we get that $(R/J(R)) *_{\alpha,w} G$ is semisimple. Thus, $(R *_{\alpha,w} G)/J(R *_{\alpha,w} G)$ is semisimple. So, $(R *_{\alpha,w} G)$ is semilocal. \square

Now, we study the semilocal property for partial crossed products when the group is not necessarily in the class \mathcal{H} .

Let G be a group and $\alpha = (\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G}, \{w_{g,h}\}_{(g,h) \in G \times G})$ a twisted partial action of a group G on a semiprime ring R . From [3], α can be extended to a twisted partial action $\alpha^* = (\{D_g^*\}_{g \in G}, \{\alpha_g^*\}_{g \in G}, \{w_{g,h}^*\}_{(g,h) \in G \times G})$ of G on the left (or right) Martindale ring of quotients Q of R . If each D_g is generated by a central idempotent $1_g \in R$, then D_g^* is also generated by 1_g . Following [9], we say that α is *X-outer*, if for all $g \in G - \{1_G\}$, the set $\phi_g = \{x \in D_g^* \mid x\alpha_g(a1_{g^{-1}}) = ax, \text{ for all } a \in R\}$ is zero. The following proposition is proved in ([9], Corollary 4.10).

Proposition 30. *Let α be an unital twisted partial action of a group G on a semiprimitive ring R . If α is X-outer, then $R *_{\alpha,w} G$ is semiprimitive*

The next result generalizes ([25], Proposition 4.4).

Proposition 31. *Let α be an unital twisted partial action of a group G on a semiprimitive ring R . If $R *_{\alpha,w} G$ is semilocal and α is partial outer, then G is finite.*

Proof. By Proposition 30, we have that $R *_{\alpha,w} G$ is semiprimitive. Then $R *_{\alpha,w} G$ is artinian and by Theorem 17, we have that G is finite. \square

We finish this section with the following two results where we only consider partial actions. The next result generalizes ([24], Theorem 2)

Corollary 32. *Let \mathcal{K} be a field of characteristic zero, R a \mathcal{K} -algebra and α is partial action of a group G on R . Suppose that R admits an enveloping action (T, β) and α is of finite type. Then $R *_\alpha G$ is semilocal if and only if R is semilocal and G is finite.*

Proof. Since α is of finite type, then by Proposition 5 we have that $R *_\alpha G$ and $T *_\beta G$ are Morita equivalent. Thus, by the methods presented in ([1] Chapter 28) we have that $T *_\beta G$ is semilocal. Hence, G is finite by ([24], Theorem 2) and R is semilocal by Corollary 13. \square

A group G is said to be *torsion* if all its elements are of finite order. We finish this section with the following result that generalizes ([24], Corollary 1).

Proposition 33. *Let α be a partial action of finite type of a group G on a ring R that has enveloping action (T, β) . If $R *_\alpha G$ is semilocal then R is semilocal and G is torsion.*

Proof. If $R *_\alpha G$ is semilocal, then R is semilocal by Corollary 13. By Proposition 5 we have that $R *_\alpha G$ and $T *_\beta G$ are Morita equivalent and by the methods presented in [1] we get that $T *_\alpha G$ is semilocal. So, by ([24], Corollary 1), G is torsion. \square

3 Krull Dimension

In the previous section we studied the artinianity of the partial crossed product, and now we will study the Krull dimension, which is a measure for a module or a ring to be artinian.

We recall that the *Krull dimension* of a right R -module M , which we will be denoted by $Kdim(M_R)$, is defined to be the deviation of the lattice $\mathcal{L}_R(M)$ of R -submodules of M . The *right Krull dimension* of a ring R is defined to be the Krull dimension of the right R -module R , which we shall denote by $Kdim(R)$ if there is no possibility of misunderstanding. We note that the right modules of Krull dimension 0 are precisely non-zero right artinian

modules. Also is well-known that every right noetherian modules has Krull dimension. For more details and basic properties of Krull dimension, see [23].

During this section, unless otherwise stated, α is only a partial action of a group G on a ring R such that the ideals D_g , $g \in G$, are generated by central idempotents $1_g \in R$.

In the next two results we get some properties of the group when the partial skew group ring has Krull dimension that generalizes ([26], Proposition 1) and ([26], Lemma 6), respectively.

Proposition 34. *Let α be a partial action of a group G on R . If $R *_\alpha G$ has Krull dimension, then G satisfies ACC on finite subgroups.*

Proof. It follows from the same methods of ([26], Proposition 1) □

Proposition 35. *Let α be a partial action of a group G on R . If $R *_\alpha G$ has Krull dimension, then G satisfies ACC on normal subgroups*

Proof. Using similar methods of ([26], Lemma 6) we have the result. □

Now, we get a comparison of the Krull dimension between the $R *_\alpha G$ and R .

Proposition 36. *Let α a partial action of a finite group G on a ring R . If R has right Krull dimension, then $R *_\alpha G$ has right Krull dimension and $Kdim(R *_\alpha G) \leq Kdim(R)$.*

Proof. If R has right Krull dimension, then each ideal D_g has Krull dimension as a right R -module and this implies that each $\delta_g D_g$ has Krull dimension as a right R -module. Thus, $Kdim(D_g)_R = Kdim(\delta_g D_g)_R$. Hence, by ([23], Lemma 6.1.14), $R *_\alpha G = \bigoplus_{g \in G} \delta_g D_g$ has Krull dimension as a right R -module and

$$\begin{aligned} Kdim(R *_\alpha G)_R &= \sup \{ Kdim((\delta_g D_g)_R) \mid g \in G \} \\ &= \sup \{ Kdim((D_g)_R) \mid g \in G \} \\ &= \sup \{ Kdim(R), Kdim((D_g)_R) \mid g \in G - \{1_G\} \} \\ &= Kdim(R), \text{ because of } Kdim((D_g)_R) \leq Kdim(R). \end{aligned}$$

Note that the map $\varphi : \mathcal{L}_{R *_\alpha G}(R *_\alpha G) \longrightarrow \mathcal{L}_R(R *_\alpha G)$ defined by $\varphi(N) = N$ is strictly increasing and then, by ([23], Proposition 6.1.17), $R *_\alpha G$ has Krull dimension and $Kdim(R *_\alpha G) \leq Kdim((R *_\alpha G)_R)$.

So, $Kdim(R *_\alpha G) \leq Kdim((R *_\alpha G)_R) = Kdim(R)$. \square

Using similar methods of ([26], Theorem 2.4) we get the following.

Corollary 37. *Let α be a partial action of an abelian group G on R . If $R *_\alpha G$ has Krull dimension then G is finitely generated.*

We recall that a ring S is a *right Goldie ring*, if S has ACC on right annihilator ideals and S has finite right uniform dimension. Also by ([23], Proposition 6.3.5), any semiprime ring with right Krull dimension is a right Goldie ring.

Corollary 38. *Let α be a partial action of any group G on a semiprime ring R . If R has right Krull dimension then $R *_\alpha G$ is a right Goldie ring.*

Proof. If R is a semiprime ring with Krull dimension, then R is a semiprime right Goldie ring. Hence, by ([7], Lemma 3.6), $R *_\alpha G$ is semiprime and by Proposition 36, $R *_\alpha G$ has Krull dimension. So, $R *_\alpha G$ is a Goldie ring. \square

Now, using Krull dimension, we study the artinianity between R and $R *_\alpha G$.

Corollary 39. *Let α be a partial action of a group G on a ring R where only a finite number of the ideals D_g is not zero. Then, $R *_\alpha G$ is right artinian if and only if R is right artinian and G is finite.*

Proof. It is a direct consequence of Theorem 17 and Proposition 36, since right artinian rings are precisely the rings with right Krull dimension 0. \square

Proposition 40. *Let α be a partial action of a group G on a ring R and H a subgroup of G . If $R *_\alpha G$ has right Krull dimension, then $R *_\alpha H$ has right Krull dimension and $Kdim(R *_\alpha G) \leq Kdim(R *_\alpha H)$. In particular, if $R *_\alpha G$ has right Krull dimension, then R has right Krull dimension and $Kdim(R *_\alpha G) \geq Kdim(R)$.*

Proof. By Proposition 8 it is easy to see that the map

$$\varphi : \mathcal{L}_{R *_{\alpha_H} H}(R *_{\alpha_H} H) \longrightarrow \mathcal{L}_{R *_{\alpha} G}(R *_{\alpha} G)$$

defined by $\varphi(I) = I(R *_{\alpha} G)$ is strictly increasing. So, by ([23], Proposition 6.1.17) we have that $R *_{\alpha_H} H$ has Krull dimension and

$$Kdim(R *_{\alpha_H} H) \leq Kdim(R *_{\alpha} G).$$

□

An immediate consequence of Propositions 36 and 40 we have the following result.

Corollary 41. *Let α be a partial action of finite group G on a ring R . Then R has right Krull dimension if and only if $R *_{\alpha} G$ has right Krull dimension. In this case, $Kdim(R) = Kdim(R *_{\alpha} G)$.*

According to [26] a group G is locally finite if each finitely generated subgroup of G is finite.

Next we have a partial generalization of ([26], Corollary 2).

Theorem 42. *Let α be a partial action of a locally finite group G on a ring R . Then $R *_{\alpha} G$ has right Krull dimension if and only if R has right Krull dimension and G is finite.*

Proof. Suppose that $R *_{\alpha} G$ has right Krull dimension. Then by Proposition 40, R has Krull dimension. We claim that G is finite. In fact, suppose that G is infinite. Thus, there exists $f_i \in G$, $i \geq 1$ such that $f_i \notin \langle f_1, \dots, f_{i-1} \rangle$ and $\langle f_1 \rangle \subsetneq \langle f_1, f_2 \rangle \subsetneq \dots$ is strictly increasing which contradicts the Proposition 34. The converse follows from Proposition 36. □

4 Triangular matrix representations of partial skew group rings

In this section, we characterize the enveloping action of partial actions of groups on triangular matrix algebras and we study triangular matrix representation of partial skew group rings.

We begin with the following definition which partially generalizes ([19], Definition 2.7).

Definition 43. Let N be a (R, S) -bimodule, $\alpha_1 = \{\alpha_R^g : D_R^{g^{-1}} \rightarrow D_R^g : g \in G, D_R^g \triangleright R\}$, $\alpha_2 = \{\alpha_S^g : D_S^{g^{-1}} \rightarrow D_S^g : g \in G, D_S^g \triangleright S\}$ be partial actions of G on R and S , respectively. A partial action α of G on N relative to (α_1, α_2) if there exists a collection of (D_R^g, D_S^g) -bi-submodules N_g of N and isomorphisms of abelian groups $\alpha_g : N_{g^{-1}} \rightarrow N_g$ such that the following properties are satisfied:

- (i) $\alpha_g(r_{g^{-1}}n_{g^{-1}}) = \alpha_1^g(r_{g^{-1}})\alpha_g(n_{g^{-1}})$ and $\alpha_g(n_{g^{-1}}s_{g^{-1}}) = \alpha_g(n_{g^{-1}})\alpha_2^g(s_{g^{-1}})$, for all $r_{g^{-1}} \in D_R^{g^{-1}}$, $s_{g^{-1}} \in D_S^{g^{-1}}$ and $n_{g^{-1}} \in N_{g^{-1}}$.
- (ii) $\alpha_e = id_N$ and $N_e = N$.
- (iii) $\alpha_g(N_{g^{-1}} \cap N_h) = N_g \cap N_{gh}$, for all $g, h \in G$.
- (iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for all $x \in \alpha_{h^{-1}}(N_h \cap N_{g^{-1}})$.

In the next result, we study necessary conditions for the existence of enveloping actions of partial actions on bimodules and the proof similarly follows as in ([10], Theorem 4.5) and we put it here for reader's convenience.

Theorem 44. Let N be a (R, S) -bimodule, α a relative (α_1, α_2) -partial action of G on N , where α_1 is a partial action of G on R and α_2 is a partial action of G on S . If (R, α_1, G) and (S, α_2, G) have enveloping actions (T_1, γ, G) and (T_2, θ, G) , respectively, such that $1_g^R n = n 1_g^S = n$, for all $n \in N$, then there exists a (T_1, T_2) -bimodule M with a global action β of G on M such that the following properties are satisfied:

- (i) $\beta_g(zm) = \gamma_g(z)\beta_g(m)$ and $\beta_g(ms) = \beta_g(m)\theta_g(s)$, for all $m \in M$, $g \in G$, $z \in T_1$ and $s \in T_2$.
- (ii) N is a (T_1, T_2) -submodule of M .
- (iii) $\beta_g|_{N_{g^{-1}}} = \alpha_g$, for all $g \in G$.
- (iv) $\beta_g(N) \cap N = N_g$, for all $g \in G$.

Proof. We consider $F(G, N)$, $F(G, R)$ and $F(G, S)$ be the Cartesian product of the copies of N , R and S indexed by the elements of G , respectively, that is, the algebra of all functions of G into N , G into R and G into S . We define the actions $(fg)(s) = f(s)g(s)$ and $(hf)(s) = h(s)f(s)$, for all $s \in G$, $f \in F(G, N)$, $g \in F(G, S)$ and $h \in F(G, R)$. Note that $F(G, N)$ is a

$(F(G, R), F(G, S))$ -bimodule. As in the proof of ([10], Theorem 4.5) we have global actions β_1 and β_2 of G on $F(G, R)$ and $F(G, S)$, respectively. We define the relative (β_1, β_2) -global action β of G on $F(G, N)$ by $\beta_g(f)|_h = f(h^{-1}g)$. It is not difficult to show that β is an isomorphism. Next, we define the homomorphism of (R, S) -bimodules $\varphi : N \rightarrow F(G, N)$ by $\varphi(n)|_g = \alpha_g(1_{g^{-1}}^R n) = \alpha_g(n 1_{g^{-1}}^S)$. We clearly have that φ is injective. Let $M = \sum_{g \in G} \beta_g(\varphi(N))$. Then for each $g \in G$, we have that $\beta_g(N)$ is a $(\gamma_g(R), \theta_g(S))$ -bimodule with the actions $\gamma_g(r)\beta_g(n) = \beta_g(rn)$ and $\beta_g(n)\gamma_g(s) = \beta_g(ns)$, for all $g \in G$, $r \in R$, $s \in S$ and $n \in N$.

Since, for all $g \in G$, $\gamma_g(R)$ and $\theta_g(S)$ are ideals of T_1 and T_2 , respectively, then we easily obtain that M is a (T_1, T_2) -bimodule. The other requirements of the theorem follows by the same methods of ([10], Theorem 4.5). \square

Now, we recall the well-known definition of a triangular matrix algebra associated to a (R, S) -bimodule N .

Definition 45. *Let N be an (R, S) -bimodule. The triangular matrix algebra associated to the triple (R, N, S) is the algebra $\mathcal{L} = \{(r, n, s) : r \in R, n \in N, s \in S\}$ with usual sum and multiplication rule $(r, n, s) \cdot (r_1, n_1, s_1) = (rr_1, rn_1 + ns_1, ss_1)$.*

Remark 46. *It is convenient to point out that*

$$\mathcal{L} = \left\{ \begin{pmatrix} r & n \\ 0 & s \end{pmatrix} : r \in R, n \in N, s \in S \right\}$$

From now on we denote the triangular matrix algebra associated to the triple (R, N, S) as $\mathcal{L} = (R, N, S)$.

The next result is probably well-known, but we could not find a proper reference and we will give a sketch of the proof.

Lemma 47. *Let $\mathcal{L} = (R, N, S)$ be a triangular matrix algebra, where N is (R, S) -bimodule. Then for any ideal J of \mathcal{L} is of the form $J = (J_1, N_2, J_3)$, where J_1 is an ideal of R , J_3 is an ideal of S and N_2 is an (R, S) -bimodule. Moreover, if J is generated by a central idempotent, then J_1 and J_3 are generated by central idempotents.*

Proof. We define

$$\begin{aligned}
J_1 &= \left\{ a \in R \mid \exists \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in J \right\}, \\
N_2 &= \left\{ n \in N \mid \exists \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \in J \right\}, \\
J_3 &= \left\{ c \in S \mid \exists \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \in J \right\}.
\end{aligned}$$

We clearly have that J_1 and J_3 are ideals of R and S and N_2 is an (R, S) -bimodule. We easily have that $J = (J_1, N_2, J_3)$. Note that each idempotent in \mathcal{L} is of the form $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$, where e and f are idempotents of R and S , respectively. The result follows. \square

The following result gives a characterization of the existence of enveloping actions of partial actions of groups on triangular matrix algebras.

Theorem 48. *Let $\mathcal{L} = (R, N, S)$ where R and S are rings with identity, N a (R, S) -bimodule and α a partial action of a group G on \mathcal{L} , where $\alpha = \{\alpha_g : T_{g^{-1}} \rightarrow T_g : g \in G\}$. Then (\mathcal{L}, α) has enveloping action (T, β) such that for each $g \in G$ $\alpha_g(1_{g^{-1}}^R, 0, 0) = (1_g^R, 0, 0)$ and $\alpha_g(0, 0, 1_{g^{-1}}^S) = (0, 0, 1_g^S)$ if and only if there exists partial actions α_1 of G on R , α_3 of G on S and a relative partial (α_1, α_3) -action α_2 of G on N such that (R, α_1) , (N, α_2) and (S, α_3) have enveloping actions. Moreover, T is a triangular matrix algebra.*

Proof. Suppose that (\mathcal{L}, α) has enveloping action (T, β) . We only construct the partial action for the ring R , because the other constructions are similar. By ([10], Theorem 4.5) each ideal T_g , $g \in G$, is generated by a central idempotent and by Lemma 47, each ideal T_g of \mathcal{L} associated to the partial action α of G is of the form $T_g = (R_g, N_g, S_g)$, where N_g is a (R, S) -sub-bimodule of N and R_g and S_g are generated by central idempotents 1_g^R and 1_g^S , respectively. For each $g \in G$, we consider $A_g = \beta_g(R, 0, 0)$. We claim that $A_g \cap (R, 0, 0) = (R_g, 0, 0)$. In fact, let $y \in A_g \cap (R, 0, 0)$. Since $y = (a', b', c')$, then we obtain that $y = y(1_R, 0, 0) = (a', 0, 0)$. Thus, $y \in (R_g, 0, 0)$.

On the other hand, let $(a, 0, 0) \in (R_g, 0, 0)$. Then, we have that $(a, 0, 0) = \beta_g(r, n, s)$. By the fact that $\beta(1_{g^{-1}}^R, 0, 0) = (1_g^R, 0, 0)$ we have that $(a, 0, 0) = (a, 0, 0)(1_g, 0, 0) = \beta_g(r, n, s)\beta_g(1_{g^{-1}}, 0, 0) = \beta_g(r1_{g^{-1}}, 0, 0) \in \beta_g(R, 0, 0)$.

Note that $\alpha_g(R_{g^{-1}}, 0, 0) = \beta_g(R_{g^{-1}}, 0, 0) = (R_g, 0, 0)$. Thus, for each $g \in G$ we consider the ideals R_g and we define the isomorphisms $\alpha_g^R : R_{g^{-1}} \rightarrow R_g$ by $\alpha_g^R = \pi_g \circ \alpha_g \circ i_g$, where $\pi_g : (R_g, 0, 0) \rightarrow R_g$ is the natural projection and $i_g : R_g \rightarrow (R_g, 0, 0)$ is the natural inclusion. Using the fact that α is a partial action of G on \mathcal{L} we easily obtain that $\alpha_1 = \{\alpha_g^R : R_{g^{-1}} \rightarrow R_g\}$ is a partial action of G on R .

Conversely, by assumption and ([10], Theorem 4.5) we have that (R, α_1) and (S, α_3) have enveloping actions (T_1, β_1) and (T_3, β_3) and by Theorem 42, (N, α_2) has enveloping action (M, β_2) . We define a global action of G on (T_1, M, T_3) as follows: for each $g \in G$, $\gamma_g(a, b, c) = (\beta_1^g(a), \beta_2^g(b), \beta_3^g(c))$. It is standard to show that $Q = \sum_{g \in G} \gamma_g(R, N, S)$ with the action defined as above is an enveloping action for (\mathcal{L}, α) and Q is a triangular matrix algebra. \square

Lemma 49. *Let $\theta : (R, N, S) \rightarrow (R', N', S')$ be an isomorphism of algebras where N is a (R, S) -bimodule, N' is a (R', S') -bimodule and, R and S are not necessarily unital rings. The following conditions are equivalent:*

- (a) $\theta(r, n, s) = (\theta_1(r), \theta_2(n), \theta_3(s))$, where $\theta_1 : R \rightarrow R'$ and $\theta_3 : S \rightarrow S'$ are isomorphisms of rings and $\theta_2 : N \rightarrow N'$ is a (R, S) -bimodule isomorphism.
- (b) $\theta(R, 0, 0) \subseteq (R', 0, 0)$ and $\theta(0, 0, S) \subseteq (0, 0, S')$.

Proof. The proof that (a) implies (b) is standard. For the converse, we define $\theta_1 : R \rightarrow R'$, $\theta_2 : N \rightarrow N'$ and $\theta_3 : S \rightarrow S'$ by $(\theta_1(r), 0, 0) = \theta(r, 0, 0)$, $(0, \theta_2(n), 0) = \theta(0, n, 0)$ and $(0, 0, \theta_3(s)) = \theta(0, 0, s)$. It is standard to show that θ_1 and θ_3 are homomorphism of rings. Note that $(0, \theta_2(rn), 0) = \theta(0, rn, 0) = \theta((r, 0, 0)(0, n, 0)) = (\theta_1(r), 0, 0)(0, \theta_2(n), 0) = (0, \theta_1(r)\theta_2(n), 0)$. Thus, $\theta_2(rn) = \theta_1(r)\theta_2(n)$ and by similar methods we show that $\theta_2(ns) = \theta_2(n)\theta_3(s)$. Moreover, by the fact that θ is an isomorphism, we have that θ_1 , θ_2 and θ_3 are isomorphisms. \square

The following result under certain conditions, we characterize the partial skew group rings over a triangular matrix algebras, i.e, a triangular matrix representation of partial skew group rings.

Theorem 50. *Let $\mathcal{L} = (R, N, S)$ and $\alpha = \{\alpha_g : D_{g^{-1}} \rightarrow D_g, g \in G\}$ a partial action of G on \mathcal{L} . Suppose that for each $g \in G$, the ideals D_g are generated by central idempotents. Then there exists partial actions α_1, α_3*

of G on R and S , a relative (α_1, α_3) -partial action α_2 of G on N such that $\mathcal{L} *_{\alpha} G \simeq (R *_{\alpha_1} G, M, S *_{\alpha_3} G)$, where M is a $(R *_{\alpha_1} G, S *_{\alpha_3} G)$ -bimodule.

Proof. By assumption, Lemmas 47 and 49 and Theorem 48, for each $g \in G$, we have the ideals R_g, S_g of R and S , submodules N_g of N with isomorphisms

$$\alpha_1^g : R_{g^{-1}} \rightarrow R_g,$$

$\alpha_2^g : N_{g^{-1}} \rightarrow N_g$ and $\alpha_3^g : S_{g^{-1}} \rightarrow S_g$, and we have the partial actions α_1 and α_3 of G on R and S and α_2 a relative (α_1, α_3) -partial action of G on N . We claim that $\mathcal{L} *_{\alpha} G \simeq (R *_{\alpha_1} G, M, S *_{\alpha_3} G)$, where M is a $(R *_{\alpha_1} G, S *_{\alpha_3} G)$ -bimodule. In fact, by Theorem 44 and ([10], Theorem 4.5) we have that (\mathcal{L}, α) , (R, α_1) , (N, α_2) and (S, α_3) have enveloping actions (T, β) , (T_1, β^1) , (T_2, β^2) and (T_3, β^3) , respectively. By similar methods presented in [19] we have that $T *_{\beta} G = (T_1 *_{\beta^1} G, T_2 *_{\beta^2} G, T_3 *_{\beta^3} G)$, where $T_2 *_{\beta^2} G$ is a $(T_1 *_{\beta^1} G, T_3 *_{\beta^3} G)$ -bimodule whose elements are the finite sums $\sum_{g \in G} a_g \delta_g$ with usual sum and multiplication rule is $(s_h \delta_h)(a_g \delta_g) = s \beta_h^2(a_g) \delta_{hg}$ and $(a_g \delta_g)(w \delta_h) = a_g \beta_g^3(w) \delta_{gh}$, for all $s_h \delta_h \in T_1 *_{\beta^1} G$, $a_g \delta_g \in T_2 *_{\beta^2} G$ and $w \delta_h \in T_3 *_{\beta^3} G$. We consider $M = \{f \in T_2 *_{\beta^2} G : (0, f, 0) \in \mathcal{L} *_{\alpha} G\}$ and let

$$y \in (R *_{\alpha_1} G, M, S *_{\alpha_3} G).$$

Then $y = (\sum_{g \in G} r_g \delta_g, \sum_{g \in G} n_g \delta_g, \sum_{g \in G} s_g \delta_g) = (\sum_{g \in G} r_g \delta_g, 0, 0) + (0, \sum_{g \in G} n_g \delta_g, 0) + (0, 0, \sum_{g \in G} s_g \delta_g) = \sum_{g \in G} (r_g, 0, 0) \delta_g + \sum_{g \in G} (0, n_g, 0) \delta_g + \sum_{g \in G} (0, 0, s_g) \delta_g \in \mathcal{L} *_{\alpha} G$.

On the other hand, for each $(r_g, s_g, n_g) \delta_g \in \mathcal{L} *_{\alpha} G$ we have that $(r_g, n_g, s_g) \delta_g = (r_g \delta_g, n_g \delta_g, s_g \delta_g) \in (R *_{\alpha_1} G, M, S *_{\alpha_3} G)$, since $n_g \delta_g \in M$ because of $n_g \delta_g \in T_1 *_{\beta^1} G$ and $(0, n_g \delta_g, 0) \in \mathcal{L} *_{\alpha} G$. So, the result follows. \square

As an immediate consequence we have the following result.

Corollary 51. *Let α be a partial action of a group G on R . Then the partial action α extends to a partial action $\bar{\alpha}$ on $\mathcal{L} = (R, R, R)$ and $\mathcal{L} *_{\bar{\alpha}} G = (R *_{\alpha} G, R *_{\alpha} G, R *_{\alpha} G)$.*

5 Dimensions of Crossed Products

In this section, we study some homological dimensions of partial crossed products. Moreover, we give some characterizations for the partial crossed products to be symmetric and Frobenius algebras. We begin with the following proposition.

Proposition 52. *Let R be a K -algebra, where K is a commutative ring, α an unital twisted partial action of a finite group G on R such that $|G|$ a unit in R and M a left $R *_{\alpha,w} G$ -module. If N is a submodule of M such that N is a direct summand of M as R -module, then N is a direct summand as $R *_{\alpha,w} G$ -module.*

Proof. Let $\pi : M \rightarrow N$ be the natural projection as R -module. We define $\Psi : M \rightarrow N$ by $\Psi(v) = \frac{1}{|G|} \sum_{g \in G} w_{g^{-1},g}^{-1} 1_{g^{-1}} \delta_{g^{-1}} \pi(1_g \delta_g v)$. It is not difficult to see that Ψ is an homomorphism of left $R *_{\alpha,w} G$ -modules and $\Psi(\alpha) = \alpha$, for all $\alpha \in N$. So, the result follows. \square

The following definition is well-known.

Definition 53. *Let P be an R -module. We say that P is projective if and only if for every surjective module homomorphism $f : N \rightarrow M$ and every module homomorphism $g : P \rightarrow M$, there exists a homomorphism $h : P \rightarrow N$ such that $fh = g$.*

Proposition 54. *Let α be an unital twisted partial action of a group G on a ring R and P a left $R *_{\alpha,w} G$ -module. If P is projective as left $R *_{\alpha,w} G$ -module, then P is projective as left R -module.*

Proof. Let $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ be an exact sequence of $R *_{\alpha,w} G$ -modules such that F is free. Since M is projective as R -module, then this sequence splits as R -modules. Hence, by Proposition 52 we have that this sequence splits as $R *_{\alpha,w} G$ -modules. So, M is a projective as left $R *_{\alpha,w} G$ -module. \square

A projective resolution $0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_0 \rightarrow M \rightarrow 0$ of the left R -module M is said to be of length n , where X_i are projectives R -module for all $i \in \{0, \dots, n\}$. The smallest such n is called the projective dimension of M , denoted by $pd_R M$ (if M has no finite projective resolution, we set $pd_R M = \infty$.) In this case the left global dimension of R is $sup = \{pd_R M : M \text{ is a left } R\text{-module}\}$.

Theorem 55. *Let α be an unital twisted partial action of a finite group G on a K -algebra R such that $|G|^{-1} \in R$. Then $lgdim R = lgdim(R *_{\alpha,w} G)$.*

Proof. By Proposition 54 we have that $lgdim R *_{\alpha,w} G \leq lgdim R$. Using ([23], 7.2.8) we have that $lgldim R \leq lgldim(R *_{\alpha,w} G) + pd(R *_{\alpha,w} G)$. Since $R *_{\alpha,w} G$ is a left free R -module, then $pd(R *_{\alpha,w} G) = 0$. So, $lgdim(R) \leq lgdim(R *_{\alpha,w} G)$. Therefore, $lgdim(R) = lgdim(R *_{\alpha,w} G)$ \square

According to [23], a ring S is said to be hereditary if all S -modules have projective resolution of length at most 1. Moreover, a ring S is said to be semi-simple if for any left ideal of R is a direct summand of R as left R -module. By ([23], 7.2.7) any semi-simple ring has left global dimension zero and any hereditary algebra has left global dimension 1.

The proof of the following result is direct consequence of the last theorem.

Corollary 56. *With the same assumptions of Theorem 55, the following statements hold.*

- (i) *R is hereditary if and only if $R *_{\alpha,w} G$ is hereditary.*
- (ii) *R is a semisimple artinian ring if and only if $R *_{\alpha,w} G$ is a semisimple artinian ring.*

To study the weak global dimension we need the following three definitions that appears in [23].

Definition 57. *Let M be a right R -module. Then M is said to be flat if $M \otimes_R$ is an exact functor.*

Definition 58. *Let M be a right R -module. The flat dimension fdM_R of the module M is defined as the shortest length of a flat resolution of M_R*

$$0 \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

where $F_i, i \in \{1, \dots, n\}$ are flat modules.

Definition 59. *The weak global dimension of a ring R is $w.dim R = \sup\{fdM : M \text{ is a right } R\text{-module}\}$*

Theorem 60. *Let α be an unital twisted partial action of a group G on R such that $|G|$ is a unit in R . Then $w.dim(R *_{\alpha,w} G) = w.dim(R)$.*

Proof. By ([23], Lemma 7.2.2) all flat left $R *_{\alpha, w} G$ -modules are flat R -modules and we get that $w.\dim R *_{\alpha, w} G \leq w.\dim R$. Since by ([23], 7.2.8),

$$w.\dim R \leq w.\dim R *_{\alpha, w} G + f d_R R *_{\alpha} G$$

and $R *_{\alpha, w} G$ is a flat R -module because of being a free R -module, then we have that $w.\dim R \leq w.\dim R *_{\alpha, w} G$ \square

According to ([29], Theorem IV.2.1) a finite dimensional K algebra R is said to be a Frobenius algebra if there exists a K -linear form $\varphi : R \rightarrow K$ such that $\ker \varphi$ does not contain a nonzero left ideal of R .

Theorem 61. *Let α be an unital twisted partial action of a finite group G on a finite dimensional K -algebra R , where K is a field. If R is Frobenius, then $R *_{\alpha, w} G$ is Frobenius.*

Proof. We consider the natural projection $\pi : R *_{\alpha, w} G \rightarrow R$ that is an homomorphism of left R -modules. By assumption there exists a nonzero form $f : R \rightarrow K$ and we get $f \circ \pi$ is a nonzero form. Let I be a nonzero left ideal of $R *_{\alpha, w} G$ that is contained in $\ker(f \circ \pi)$. Thus, there exists a nonzero element $\eta = \sum a_g \delta_g \in I$ and we may assume that $\eta = a_e + \sum_{g \neq e} a_g \delta_g$. Hence, $0 \neq \pi(I) \neq \ker f$ which contradicts the fact that R is Frobenius. So, $R *_{\alpha, w} G$ is a Frobenius algebra. \square

According to ([29], Theorem 2.2) a finite dimensional K -algebra R is a symmetric algebra if there exists K -linear form $\varphi : R \rightarrow K$ such that $\varphi(ab) = \varphi(ba)$, for all $a, b \in R$ and $\ker \varphi$ does not contain a nonzero one-sided ideal of R .

The proof of the following result follows the same ideas of Theorem 61.

Theorem 62. *Let α be an unital twisted partial action of a finite group G on a finite dimensional K -algebra R , where K is a field. If R is symmetric, then $R *_{\alpha, w} G$ is symmetric.*

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